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Convolution of Ultradistributions and Field Theory

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A general definition of convolution between two arbitrary tempered ultradistributions is given. When one of the tempered ultradistributions is rapidly decreasing this definition coincides with the definition of J. Sebastiao e Silva. The product of two arbitrary distributions of exponential type is defined via the convolution of its corresponding Fourier transforms. Several examples of convolution of two tempered ultradistributions and singular products are given. In particular, we reproduce the results obtained by A. Gonzales Dominguez and A. Bredimas.

1. INTRODUCTION

In physics, it is sometimes necessary to work with functions that grow exponentially in space or time. For those cases the Schwartz space of tempered distributions [1] are too restrictive. On the other hand, the space of test functions with bounded support allows the distributions to blow up more rapidly than any exponential. In this sense they should be considered to be too "permissive" for physical applications. What is needed is an equilibrium between the necessities in x space and the possibility to work in the Fourier transformed space (p space) with propagators. From a mathematical point of view the latter are analytic functionals defined on a space of entire test functions.

We shall see that a point of equilibrium is achieved by working with tempered ultradistributions (see below). They also have the advantage of being representable by means of analytic functions. Thus in general they are easier to work with and have interesting properties. One of those properties,

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as we shall see, is the possibility of defining a convolution product which is general enough to be valid for any two tempered ultradistributions, and of course, this automatically provides a definition for the product of distributions of the exponential type in x space.

In Sections 2 and 3 we define the distributions of exponential type and the Fourier-transformed tempered ultradistributions. Each of them is part of a Gelfand triplet (or rigged Hilbert space [2, 3]) together with their respectives duals and a "middle term" Hilbert space. In Section 4 we give a general expression for the convolution. We also state and prove some existence teorems. In Section 5 we present several examples. Some of them imply singular products. Finally, in section 6 we discuss of the principal results. For the benefit of the reader an Appendix is added containing some formulas utilized in the text.

2. DISTRIBUTIONS OF EXPONENTIAL TYPE

For the sake of the reader we present a brief description of the principal properties of tempered ultradistributions.

The space H of test functions that $e^{p|x|}\left|D^q\phi(x)\right|$ is bounded for any p and q is defined [4] by means of the countable set of norms:

$$\|\hat{\phi}\|_p'' = \sup_{0\le q\le p,x} e^{p|x|}\left|D^q\hat{\phi}(x\right|, \qquad p = 0, 1, 2, \ldots \tag{2.1}$$

According to ref. 5, H is a $\mathcal{K}\{M_p\}$ space with

$$M_p(x) = e^{(p-1)|x|}, \qquad p = 1, 2, \ldots \tag{2.2}$$

$\mathcal{K}\{e^{(p-1)|x|}\}$ satisfies condition ($\mathcal{N}$) of Gelfand [2]. It is a countable Hilbert and nuclear space:

$$\mathcal{K}\{e^{(p-1)|x|}\} = H = \bigcap_{p=1}^{\infty} H_p \tag{2.3}$$

where H_p is obtained by completing H with the norm induced by the scalar product:

$$\langle\hat{\phi}, \hat{\psi}\rangle_p = \int_{-\infty}^{\infty} e^{2(p-1)|x|} \sum_{q=0}^{p} D^q\overline{\hat{\phi}}(x)\, D^q\hat{\psi}(x)\, dx; \qquad p = 1, 2, \ldots \tag{2.4}$$

If we take the usual scalar product

$$\langle\hat{\phi}, \hat{\psi}\rangle = \int_{-\infty}^{\infty} \overline{\hat{\phi}}(x)\hat{\psi}(x) \tag{2.5}$$

then H, completed with (2.5), is the Hilbert space $\mathcal{H}$ of square-integrable functions.

The space of continuous linear functionals defined on H is the space Λ_∞ of the distributions of exponential type [4].

The "nested space"

$$(H, \mathcal{H}, \Lambda_\infty) \tag{2.6}$$

is a Gelfand triplet (or a rigged Hilbert space [2, 3].

Any Gelfand triplet $(\mathcal{A}, \mathcal{H}, \mathcal{A}')$ has the fundamental property that a linear and symmetric operator on $\mathcal{A}$ admitting an extension to a self-adjoint operator in $\mathcal{H}$ has a complete set of generalized eigenfunctions in $\mathcal{A}'$ with real eigenvalues.

3. TEMPERED ULTRADISTRIBUTIONS

The Fourier transform of a function $\hat{\phi} \in H$ is

$$\phi(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx\, e^{izx}\, \phi(x) \tag{3.1}$$

$\phi(z)$ is entire analytic and rapidly decreasing on straight lines parallel to the real axis. We shall call h the set of all such functions:

$$h = \mathcal{F}\{H\} \tag{3.2}$$

It is a $\mathcal{Z}\{M_p\}$ space [2, 3] countably normed and complete, with

$$M_p(z) = (1 + |z|)^p \tag{3.3}$$

h is also a nuclear space with norms

$$\|\phi\|_{pn} \sup_{|\mathrm{Im}(z)| \leq n} (1 + |z|)^p |\phi(z)| \tag{3.4}$$

We can define the usual scalar product

$$\langle \phi(z), \psi(z) \rangle = \int_{-\infty}^{\infty} \phi(z)\psi_1(z)\, dz = \int_{-\infty}^{\infty} \overline{\hat{\phi}}(x)\hat{\psi}(x)\, dx \tag{3.5}$$

where

$$\psi_1(z) = \int_{-\infty}^{\infty} dx\, e^{-izx}\, \hat{\psi}(x)$$

By completing h with the norm induced by (3.5) we get the Hilbert space of square-integrable functions.

The dual of h is the space $\mathcal{U}$ of tempered ultradistributions [4]. In other words, a tempered ultradistribution is a continuous linear functional defined on the space h of entire functions rapidly decreasing on straight lines parallel to the real axis.

The set $(h, \mathcal{H}, \mathcal{U})$ is also a Gelfand triplet.

$\mathcal{U}$ can also be characterized in the following way [4]: let $\mathcal{A}$ be the space of all functions $F(z)$ such that:

I. $F(z)$ is analytic for $\{z \in \mathcal{C}: |\mathrm{Im}(z)| > p\}$.

II. $F(z)/z^p$ is bounded continuous in $\{z \in \mathcal{C}: |\mathrm{Im}(z)| \geq p\}$, where $p = 0, 1, 2, \ldots$ depends on $F(z)$.

Let Π be the set of all z-dependent polynomials, $z \in \mathcal{C}$. Then $\mathcal{U}$ is the quotient space:

III. $\mathcal{U} = \mathcal{A}/\Pi$.

Due to these properties it is possible to represent any ultradistribution as [4]

$$F(\phi) = \langle F(z), \phi(z) \rangle = \oint_\Gamma dz\, F(z)\phi(z) \tag{3.6}$$

where the path Γ runs parallel to the real axis from $-\infty$ to ∞ for $\mathrm{Im}(z) > \rho$, $\rho > p$, and back from ∞ to $-\infty$ for $\mathrm{Im}(z) < -\rho$, $-\rho < -p$. [Γ lies outside a horizontal band of width $2p$ containing all the singularities of $F(z)$.]

Formula (3.6) will be our fundamental representation for a tempered ultradistribution. Sometimes use will be made of "Dirac's formula" for ultradistributions [6]:

$$F(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} dt\, \frac{f(t)}{t - z} \tag{3.7}$$

where the "density" $f(t)$ is such that

$$\oint_\Gamma dz\, F(z)\phi(z) = \int_{-\infty}^{\infty} dt\, f(t)\phi(t) \tag{3.8}$$

While $F(z)$ is analytic on Γ, the density $f(t)$ is in general singular, so that the r.h.s. of (3.8) should be interpreted in the sense of distribution theory.

Another important property of the analytic representation is the fact that on Γ, $F(z)$ is bounded by a power of z [4]:

$$|F(z)| \leq C|z|^p \tag{3.9}$$

where C and p depend on F.

The representation (3.6) makes evident that the addition of a polynomial $P(z)$ to $F(z)$ does not alter the ultradistribution:

$$\oint_{\Gamma} dz\,\{F(z) + P(z)\}\phi(z) = \oint_{\Gamma} dz\,F(z)\phi(z) + \oint_{\Gamma} dz\,P(z)\phi(z)$$

But

$$\oint_{\Gamma} dz\,P(z)\phi(z) = 0$$

as $P(z)\phi(z)$ is entire analytic (and rapidly decreasing), therefore

$$\oint_{\Gamma} dz\,\{F(z) + P(z)\}\phi(z) = \oint_{\Gamma} dz\,F(z)\phi(z) \tag{3.10}$$

4. THE CONVOLUTION

If we try to define the convolution product by means of the natural formula

$$(F * G)\{\phi\} = \oint_{\Gamma_1}\oint_{\Gamma_2} dk_1\,dk_2\,F(k_1)G(k_2)\phi(k_1 + k_2) \tag{4.1}$$

we soon discover that it is not always defined. The reason is simple. The result of

$$\oint_{\Gamma} dk\,F(k)\phi(k + k') = \chi(k')$$

does not, in general, belong to h. However, if at least one of the ultradistributions F and G is rapidly decreasing (say G), then a convolution can be defined [6] by

$$H(k) = \int_{-\infty}^{\infty} dt\,f(t)G(k - t) \tag{4.2}$$

where $f(t)$ is the density associated to $F(k)$ [cf. (3.7)].

In order to eliminate the test function from (4.1) use can be made of the complex δ-function, which is an ultradistribution (Cauchy's theorem)

$$\delta_{z'}\{\phi\} = -\frac{1}{2\pi i}\oint_{\Gamma} dz\,\frac{\phi(z)}{z - z'} = \phi(z') \tag{4.3}$$

where the point z' is enclosed by Γ (this procedure was used in ref. 7). We can then write (4.1) as

$$(F * G)\{\phi\} = -\frac{1}{2\pi i}\oint_{\Gamma} dz \oint_{\Gamma_1}\oint_{\Gamma_2} dk_1\,dk_2\,\frac{F(k_1)G(k_2)}{z - k_1 - k_2}\,\phi(z) \tag{4.4}$$

The path Γ must have

$$\left|\mathrm{Im}(z)\right| > \left|\mathrm{Im}(k_1)\right| + \left|\mathrm{Im}(k_2)\right| \tag{4.5}$$

in order to embrace the point $k_1 + k_2$ ($k_1 \in \Gamma_1$, $k_2 \in \Gamma_2$).

Equation (4.4) leads to

$$F * G = H \doteq -\frac{1}{2\pi i}\oint_{\Gamma_1}\oint_{\Gamma_2} dk_1\, dk_2\, \frac{F(k_1)G(k_2)}{z - k_1 - k_2} \tag{4.6}$$

However, we do not expect (4.6) to define a tempered ultradistribution for every pair F, G. Note that in (4.1) F and G operate on $\phi(k)$, which is rapidly decreasing, while in (4.6) they act on $(z - k)^{-1}$, ($k = k_1 + k_2$). Furthermore, due to (4.5) and the fact that Γ_1 and Γ_2 run outside a horizontal band containing all the singularities of F and G, the integrand in (4.6) is analytic at every point of the integration paths. Taking into account the property (3.9) of tempered ultradistributions, we come to the conclusion that the integrations in (4.6) have at most a tempered singularity for $k \to \infty$. In order to control this possible singularity we introduce a regulator (see ref. 8).

We define

$$H_\lambda(z) = \frac{i}{2\pi}\oint_{\Gamma_1}\oint_{\Gamma_2} dk_1\, dk_2\, \frac{k_1^\lambda F(k_1) k_2^\lambda G(k_2)}{z - k_1 - k_2} \tag{4.7}$$

Now, if we have the bounds

$$\left|F(k_1)\right| \le C_1\left|k_1\right|^m, \qquad \left|G(k_2)\right| \le C_2\left|k_2\right|^n \tag{4.8}$$

Then (4.7) is convergent for

$$\mathrm{Re}(\lambda) < -l - 1; \qquad l = \max\{m, n\} \tag{4.9}$$

It is also analytic in the region (4.9) of the λ plane, as the derivative with respect to λ merely multiplies by a logarithmic factor the integrand of (4.7) without spoiling the convergence.

According to the method of ref. 8, H_λ can be analytically continued to other parts of the λ plane. In particular near the origin we have the Laurent (or Taylor) expansion

$$H_\lambda = \sum_n H^{(n)}(z)\lambda^n \tag{4.10}$$

where the sum might have terms with negative n. We now define the convolution product as the λ-independent term of (4.10):

$$H(z) = H^{(0)}(z) \tag{4.11}$$

Note that the derivatives of $H_\lambda(z)$ with respect to z can be obtained from (4.7) by taking different powers of the denominator:

$$\frac{d^p H_\lambda(z)}{dz^p} = (-1)^p p! \frac{i}{2\pi} \oint_{\Gamma_1} \oint_{\Gamma_2} dk_1\, dk_2 \frac{k_1^\Lambda F(k_1) k_2^\Lambda G(k_2)}{(z - k_1 - k_2)^p} \tag{4.12}$$

The convergence of (4.7) also ensures that of (4.12), and therefore also ensures analyticity in z outside the horizontal band defined by (4.5). We will now show that $|H_\lambda(z)|$ is bonded by a power of $|z|$ [cf. (3.9)].

To that aim we take

$$\text{Im}(\lambda) = 0; \qquad \lambda < -l - 1; \qquad z = x + iy$$

$$k_i = \kappa_i \pm i\sigma_i; \qquad \sigma_i > 0; \qquad dk_i = d\kappa_i$$

The integrals along Γ_i can be expressed as integrals on $d\kappa_i$ between $0 \to \infty$. Then we have

$$\begin{aligned} |H_\lambda| &= \frac{1}{2\pi} \left| \oint_{\Gamma_1} \oint_{\Gamma_2} dk_1\, dk_2 \frac{k_1^\lambda F(k_1) k_2^\lambda G(k_2)}{z - k_1 - k_2} \right| \\ &\leq \frac{1}{2\pi} \oint_{\Gamma_1} \oint_{\Gamma_2} \text{sgn Im}(k_1)\, dk_1\, \text{sgn Im}(k_2)\, dk_2 \frac{|k_1|^\lambda C_1 |k_1|^m |k_2|^\lambda C_2 |k_2|^n}{|z - k_1 - k_2|} \\ &\leq \frac{C_1 C_2}{2\pi} \oint_{\Gamma_1} \oint_{\Gamma_2} \text{sgn Im}(k_1)\, dk_1\, \text{sgn Im}(k_2)\, dk_2\, |k_1|^{\lambda+m} |k_2|^{\lambda+n} \\ &= \frac{8 C_1 C_2}{\pi} \int_0^\infty \int_0^\infty d\kappa_1\, d\kappa_2\, (\kappa_1^2 + \sigma_1^2)^{(\lambda+m)/2} (\kappa_2^2 + \sigma_2^2)^{(\lambda+n)/2} \end{aligned} \tag{4.13}$$

We now make the change of variables $w_i = \kappa_i^2$ and obtain

$$\begin{aligned} (4.13) &= \frac{2C_1C_2}{\pi} \int_0^\infty dw_1\, w_1^{-1/2} (w_1 + \sigma_1^2)^{(\lambda+w)/2} \\ &\quad \times \int_0^\infty dw_2\, w_2^{-1/2} (w_2 + \sigma_2^2)^{(\lambda+m)/2} \end{aligned} \tag{4.14}$$

$$\begin{aligned} &= \frac{2C_1C_2}{\pi} \mathcal{B}\left(\frac{1}{2}, -\frac{\lambda + m + 1}{2}\right) \mathcal{B}\left(\frac{1}{2}, -\frac{\lambda + n + 1}{2}\right) \\ &\quad \times \sigma_1^{(\lambda+m+1)/2} \sigma_2^{(\lambda+n+1)/2} \leq C(\lambda, m, n) |z|^{\lambda+m+n+1} \end{aligned} \tag{4.15}$$

where $\mathcal{B}(x, y)$ is the Gauss beta function.

It is to be noted that if $G(k)$ is a rapidly decreasing ultradistribution, then $H_\lambda(z)$ [Eq. (4.7)] coincides with $H_0(z)$:

$$H_0(z) = \frac{i}{2\pi} \oint_{\Gamma_1} dk_1\, F(k_1) \oint_{\Gamma_2} dk_2 \frac{G(k_2)}{z - k_1 - k_2} \tag{4.16}$$

In fact, near $\lambda = 0$ we have ($|k| > 1$)

$$\left|k^{\lambda} - 1\right| \leq \lambda(2\pi + \left|\ln|k|\right|)|k|^{\lambda} \tag{4.17}$$

$$H_{\lambda} - H_0(z) = \frac{i}{2\pi} \oint_{\Gamma_1} dk_1\, k_1^{\lambda} F(k_1) \oint_{\Gamma_2} dk_2\, (k_2^{\lambda} - 1) \frac{G(k_2)}{z - k_1 - k_2}$$
$$+ \frac{i}{2\pi} \oint_{\Gamma_1} dk_1\, (k_1^{\lambda} - 1)\, F(k_1) \oint_{\Gamma_2} dk_2 \frac{G(k_2)}{z - k_1 - k_2} \tag{4.18}$$

In Eq. (4.18) the integrals are convergent, as $G(k)$ and $k^{\lambda}G(k)$ are both rapidly decreasing. Furthermore, due to (4.17) the difference $H_{\lambda} - H_0$ is proportional to λ. Therefore

$$\lim_{\lambda \to 0} [H_{\lambda} - H_0] = 0 \tag{4.19}$$

Again, when $G(k)$ is rapidly decreasing, the convolution defined in ref. 6

$$H(z) = \int_{-\infty}^{\infty} dt\, f(t) G(z - t) \tag{4.20}$$

[where $f(t)$ is given by (3.7), (3.8)] also coincides with (4.16). To show that (4.16) implies (4.20), we use (3.8) in (4.16),

$$H_0(z) = \frac{i}{2\pi} \int_{-\infty}^{\infty} dt\, f(t) \oint_{\Gamma_2} dk_2 \frac{G(k_2)}{z - k_1 - k_2}$$

But if $G(t)$ is the density associated to $G(z)$, then

$$\frac{i}{2\pi} \oint_{\Gamma_2} dk_2 \frac{G(k_2)}{z - t - k_2} = \frac{1}{2\pi i} \int_{-\infty}^{\infty} dt_2 \frac{g(t_2)}{t_2 - (z - t)} = G(z - t)$$

i.e.,

$$H_0(z) = H(z) \tag{4.21}$$

5. EXAMPLES

In this section we are going to use definition (4.7) to evaluate the convolution of tempered ultradistributions and indirectly the product of distributions ($\in \Lambda_{\infty}$; see Section 2).

The convolution theorem states that

$$\mathcal{F}\{f_1(x) f_2(x)\} = \frac{1}{2\pi} \check{f}_1(k) * \check{f}_2(k) \tag{5.1}$$

where

$$\check{f} = \mathcal{F}\{f(x)\}(k)$$

(i) As a first example we take the distribution $x_\pm^\alpha$ (ref. 8, Chapter 1, §3.2; also ref. 9, Chapter 4), whose Fourier transform we write

$$\check{x}_\pm^\alpha = ie^{\mp i1/2\alpha}\Gamma(\alpha+1)k^{-\alpha-1}\Theta[\mp\epsilon(k)] \tag{5.2}$$

where $\Theta(x)$ is the Heaviside step function and $\epsilon(k) = \text{sgn Im}(k)$.

The ultradistribution (5.2) has a line of singularities (a discontinuity) on the real axis. Then the path Γ [cf. (2.6)] runs parallel to the real axis at a distance as small as we please, and we have

$$\mathcal{F}\{x_+^\alpha x_+^\beta\} = \frac{i}{4\pi^2}\oint_{\Gamma_1} dk_1 \oint_{\Gamma_2} dk_2 \frac{\check{x}_+^\alpha \check{x}_+^\beta}{z-k_1-k_2}$$

$$= \left[\frac{i}{4\pi^2} ie^{-i(\pi/2)\alpha}\,\Gamma(\alpha+1)ie^{-i(\pi/2)\beta}\,\Gamma(\beta+1)\right]$$

$$\times \oint_{\Gamma_1} dk_1\, k_1^{-\alpha-1}\Theta[-\epsilon(k_1)\,] \oint_{\Gamma_2} dk_2 \frac{k_2^{-\beta-1}\,\Theta[-\epsilon(k_2)]}{z-k_1-k_2}$$

The functions $\Theta[\epsilon(k_1)]$ and $\Theta[\epsilon(k_2)]$ eliminate the branches of Γ_1 and Γ_2, respectively, on the lower half-plane of k_1 and k_2. By taking the remaining integration arbitrarily close to the real axis, we get

$$\mathcal{F}\{x_+^\alpha x_+^\beta\} = -[\,]\oint_{\Gamma_1} dk_1\, k_1^{-\alpha-1}\Theta[-\epsilon(k_1)]\int_{-\infty}^{\infty} dy\, \frac{(y-i0)^{-\beta-1}}{z-k_1-y}$$

$$= -[\,]\oint_{\Gamma_1} dk_1\, k_1^{-\alpha-1}\Theta[-\epsilon(k_1)]\int_{-\infty}^{\infty} dy\, \frac{y_+^{-\beta-1}+e^{-i\pi(-\beta-1)}y_-^{-\beta-1}}{z-k_1-y}$$

$$= -[\,]\oint_{\Gamma_1} dk_1\, k_1^{-\alpha-1}\Theta[-\epsilon(k_1)]\,\frac{\Gamma(-\beta)\,\Gamma(1+\beta)}{(z-k_1)^{\beta+1}}$$

$$\times\,[e^{-i\pi(-\beta-1)} - e^{-i\pi\epsilon(z)(-\beta-1)}]$$

$$= 2i[\,]\Theta[-\epsilon(z)]\Gamma(-\beta)\,\Gamma(1+\beta)\sin\pi\,(-\beta-1)$$

$$\times \int_{\Gamma_1} dk_1 \frac{k_1^{-\alpha-1}}{(z-k_1)^{\beta+1}}\,\Theta[-\epsilon\,(k_1)]$$

$$= 2i\pi\Theta[-\epsilon(z)][\,]\int_{-\infty}^{\infty} dx\, \frac{x_+^{-\alpha-1}+e^{-i\pi(-\alpha-1)}x_-^{-\alpha-1}}{(z-x)^{\beta+1}}$$

$$= 2i\pi\Theta[-\epsilon(z)][\,]\mathcal{B}\,(-\alpha,\beta+\alpha+1)\,[e^{i\pi\epsilon(z)\alpha} - e^{i\pi\alpha}]\,z^{-\alpha-\beta-1}$$

$$= 2i\pi\{\Theta[-\epsilon(z)]\{^2 [\,]\,\frac{\Gamma(-\alpha)\Gamma(\beta+\alpha+1)}{\Gamma(\beta+1)}\,2i\sin\pi(-\alpha)\,z^{-\alpha-\beta-1}$$
$$= ie^{-i(\pi/2)\,(\alpha+\beta)}\Gamma(\alpha+\beta+1)\,z^{-\alpha-\beta-1}\Theta\,[-\epsilon(z)]$$
$$= \breve{x}_+^{\alpha+\beta} = \mathcal{F}\{x_+^{\alpha+\beta}\} = \mathcal{F}\{x_+^{\alpha}\, x_+^{\beta}\} \tag{5.3}$$

where use has been made of Eq. (A.4) of the Appendix.

For the evaluation of the convolution $\breve{x}_+^{\alpha} * \breve{x}_-^{\beta}$ the procedure is entirely similar. However, in this case one of the integrations gives rise to a factor $\Theta[-\epsilon(z)]$ and the other to a factor $\Theta[\epsilon(z)]$. Thus, instead of $\{\Theta[-\epsilon(z)]\}^2 = \Theta[-\epsilon(z)]$, we get $\Theta[-\epsilon(z)]\Theta[\epsilon(z)] = 0$. That is,

$$\breve{x}_+^{\alpha} * \breve{x}_-^{\beta} \equiv 0, \qquad \text{therefore} \qquad x_+^{\alpha}\cdot x_-^{\beta} = 0 \tag{5.4}$$

(ii) As a second example we consider Dirac δ-functions, whose Fourier transform is

$$\breve{\delta}^{(m)} = i^m k^m\,\frac{\epsilon(k)}{2} \tag{5.5}$$

For the convolution (4.7) we have

$$\breve{\delta}^{(m)} * \breve{\delta}^{(u)} = \frac{i}{4\pi}\int_{\Gamma_1} dk_1\; i^m k_1^{\lambda+m}\,\frac{\epsilon(k_1)}{2}\int_{\Gamma_2} dk_2\;\frac{i^n k_2^{\lambda+n}\epsilon(k_2)}{z-k_1-k_2}$$

(in this case, the factors ϵ_1 and ϵ_2 change the sign of the integrations of the lower half-plane of k_1 and k_2)

$$= \frac{i^{m+n+1}}{4\pi}\int_{\Gamma_1} dk_1\; k_1^{\lambda+m}\,\frac{\epsilon(k_1)}{2}\int_{-\infty}^{\infty} dy\;\frac{(y+i0)^{\lambda+n}+(y-i0)^{\lambda+n}}{z-k_1-y}$$
$$= \frac{i^{m+n+1}}{2\pi}\int_{\Gamma_1} dk_1\; k_1^{\lambda+m}\,\frac{\epsilon(k_1)}{2}\int_{-\infty}^{\infty} dy\;\frac{y_+^{\lambda+n}+\cos\pi(\lambda+n)y_-^{\lambda+n}}{z-k_1-y}$$
$$= \frac{i^{m+n+1}}{2\pi}\int_{\Gamma_1} dk_1\; k_1^{\lambda+m}\,\frac{\epsilon(k_1)}{2}\,\frac{\Gamma(\lambda+n+1)\Gamma(-\lambda-n)}{z-k_1}$$
$$\times\,[\cos\pi(\lambda+n) - e^{-i\pi\epsilon(z)(\lambda+n)}]$$
$$= -\,\frac{i\pi\epsilon(z)}{2\pi}\,i^{m+n+1}\int_{-\infty}^{\infty} dx\;\frac{x_+^{\lambda+m}+\cos\pi(\lambda+m)x_-^{\lambda+m}}{(z-x)^{-\lambda-n}}$$
$$\times\,\frac{\epsilon(z)}{2}\,i^{m+n}\,\frac{\Gamma(\lambda+m+1)\,\Gamma(-2\lambda-m-n-1)}{\Gamma(-\lambda-n)}\,z^{2\lambda+m+n+1}$$

$$= [e^{-i\pi\epsilon(z)(\lambda+ m+1)}) + \cos \pi(\lambda + m)]$$

$$= \frac{[\epsilon(z)]^2}{2} i^{m+n+1} \frac{\Gamma(\lambda + m + 1)\Gamma(-2\lambda - m - n - 1)}{\Gamma(-\lambda - n)}$$

$$\times \sin \pi(\lambda + m) z^{2\lambda+m+n+1}$$

$$\underset{\lambda\to 0}{\to} 0 = \check{\delta}^{(m)} * \check{\delta}^{(n)} \tag{5.6}$$

There are two reasons for this null result. The Γ functions have simple poles when their arguments are negative integers (or zero). So the quotient of Γ functions has a finite limit. However, they are multiplied by $\sin \pi(\lambda + m)_{\lambda\to 0} \to 0$.

Furthermore, $[\epsilon(z)]^2 = 1$, and

$$z^{2\lambda+m+n+1} \underset{\lambda\to 0}{\to} z^{m+n+1}$$

Then we can put (C = arbitrary constant)

$$\check{\delta}^{(m)} * \check{\delta}^{(n)} = C z^{m+n+1} \tag{5.7}$$

But due to property III of Section 3, the ultradistribution (5.7) is equivalent to zero.

We have then

$$\delta^{(m)}(x) \cdot \delta^{(n)}(x) = 0 \tag{5.8}$$

This result was obtained in ref. 10 and can be summarized in general as follows:

The product of two distributions with point support is zero.

(iii) We can combine examples (i) and (ii) to find the product $\delta^{(m)} \cdot \check{x}_+^\alpha$:

$$\frac{1}{2\pi} \check{\delta}^{(m)} * \check{x}_+^\alpha = \left[\frac{i}{4\pi^2} i^m i e^{-i(\pi/2)\alpha} \Gamma(\alpha + 1) \right]$$

$$\times \oint_{\Gamma_1} dk_1\, k_1^{\lambda+m} \frac{\epsilon(k_1)}{2} \oint_{\Gamma_2} dk_2 \frac{k_2^{-\alpha-1}\, \Theta[-\epsilon(k_2)]}{z - k_1 - k_2}$$

$$= 2\pi i\, \Theta[-\epsilon(z)][\ ] \int_{-\infty}^{\infty} dx \frac{x_+^{\lambda+m} + \cos \pi(\lambda + m) x_-^{\lambda+m}}{(z - x)^{\alpha+1}}$$

$$= 2\pi i \Theta[-\epsilon(z)][\ ] \frac{\Gamma(\lambda + m + 1)\Gamma(\alpha - \lambda - m)}{\Gamma(\alpha + 1)} z^{\lambda+m-\alpha}$$

$$\times [e^{-i\pi\epsilon(z)(\lambda+m+1)} + \cos \pi(\lambda + m)]$$

$$= 2\pi i \Theta[-\epsilon(z)]\left(-\frac{i^m}{4\pi^2} e^{-i(\pi/2)\alpha} \right)$$
$$\times \Gamma(\lambda + m + 1)\, \Gamma(\alpha - \lambda - m)$$
$$\times i \sin \pi(\lambda + m)\, \epsilon(z) z^{\lambda+m-\alpha} \underset{\lambda\to 0}{\to} 0 \tag{5.9}$$

if α is not an integer s such that $s \leq m$.

When $0 \leq \alpha = s \leq m$

$$\frac{1}{2\pi}\check{\delta}^{(m)} * \check{x}_+^s = -2i\pi\, \Theta[-\epsilon(z)]\frac{i^m}{4\pi^2}(-i)^s i\epsilon(z) z^{\lambda+m-s}$$
$$\times \frac{\Gamma(\lambda+m+1)}{\Gamma(\lambda+m+1-s)} \frac{\sin\pi(\lambda+m)}{\sin\pi(\lambda+m-s)}$$
$$= \frac{i^m}{2}(-i)^s \frac{\Gamma(\lambda+m+1)}{\Gamma(\lambda+m+1-s)} \frac{\sin\pi(\lambda+m)}{\sin\pi(\lambda+m-s)} \Theta[-\epsilon(z)]\epsilon(z)\, z^{\lambda+m-s}$$
$$\underset{\lambda\to 0}{\to} (-1)^s \frac{i^{m-s}}{2} \frac{m!}{(m-s)!} \frac{\epsilon(z)}{2} z^{m-s}$$
$$= \frac{(-1)^s}{2} \frac{m!}{(m-s)!} \check{\delta}^{(m-s)} \tag{5.10}$$

In particular, for $s = 0$ we get

$$\delta^{(m)}(x)\Theta(x) = \frac{1}{2}\, \delta^{(m)}(x) \tag{5.11}$$

If $\alpha = s =$ negative number $= -n$ we must be careful, as x_+^α has a pole for $\alpha = -n$. We shall deal with this case by replacing $\alpha = -n - \lambda$ in (5.9)

$$\Gamma(\alpha - \lambda - m) \to \Gamma(-2\lambda - m - n)$$
$$= -\frac{\pi}{\Gamma(2\lambda + m + n + 1)\, \sin \pi(2\lambda + m + n)}$$

and by taking the limit $\lambda \to 0$:

$$\frac{1}{2\pi}\, \check{\delta}^{(m)} * x_+^{-n} = \frac{i^{m+n}}{2} \frac{m!}{(m+n)!} \frac{(-1)^n}{2} \frac{\epsilon(z)}{2} z^{m+n} = \frac{(-1)^n}{4} \frac{m!}{(m+n)!} \check{\delta}^{(m+n)} \tag{5.12}$$

In Eqs. (5.10) and (5.12) we used

$$\Theta[-\epsilon(z)]\epsilon(z) = -\Theta[-\epsilon(z)] = \frac{1}{2}(\epsilon(z) - 1) = \frac{\epsilon(z)}{2} - \frac{1}{2}$$

$$\Theta[-\epsilon(z)]\epsilon(z)\, z^s = \frac{\epsilon(z)}{2} z^s - \frac{1}{2} z^s \approx \frac{\epsilon(z)}{2} z^s$$

since Cz^s is equivalent to zero [cf. (5.7)].

Similar expressions originate from the use of $\breve{x}_-^a$ in (5.9). In particular, if we use

$$\breve{x}^{-n} = \breve{x}^{+n} + (-1)^n \breve{x}^{-n} \tag{5.13}$$

then we easily find

$$\frac{1}{2\pi} \breve{\delta}^{(m)} * \breve{x}^{-u} = \frac{(-1)^n}{2} \frac{m!}{(m+n)!} \breve{\delta}^{(m+n)} \tag{5.14}$$

The case $m = 0$, $n = 1$ was first treated in ref. 11. For $m = n$, Eq. (5.14) agrees with ref. 12.

(iv) To illustrate the use of (4.10) and (4.11), we are now examine an interesting example. Let us take the ultradistribution (5.13), which is found to be

$$\breve{x}^{-n} = \frac{(-i)^n \pi}{(n-1)!} \left[-\frac{1}{\pi i} \ln(k) + \frac{\epsilon(k)}{2} \right] k^{n-1} \tag{5.15}$$

The convolution product is now

$$\breve{x}^{-m} * \breve{x}^{-n}$$

$$= -\frac{(-i)^{m+n+1}}{4\,(m-4)!(n-1)!} \oint_{\Gamma_1} \oint_{\Gamma_2} dk_1\, dk_2$$

$$\times \left\{ -\frac{1}{\pi^2} \frac{k_1^{\lambda+m-1} \ln(k_1)\, k_2^{\lambda+n-1} \ln(k_2)}{z - k_1 - k_2} - \frac{1}{2\pi i} \frac{k_1^{\lambda+m-1} \ln(k_1) k_2^{\lambda+n-1}\, \epsilon(k_2)}{z - k_1 - k_2} \right.$$

$$\left. + -\frac{1}{2\pi i} \frac{k_1^{\lambda+m-1}\, \epsilon(k_1)\, k_2^{\lambda+n-1} \ln(k_2)}{z - k_1 - k_2} + \frac{1}{4} \frac{k_1^{\lambda+m-1}\, \epsilon(k_1) k_2^{\lambda+n-1}\, \epsilon(k_2)}{z - k_1 - k_2} \right\} \tag{5.16}$$

The last term of (5.16) is null according to example (ii). We analyze now the first term. We use the identity

$$k^{\lambda+m-1} \ln(k) = D_\alpha k^{\alpha+m-1}; \qquad D_\alpha = \left. \frac{\partial}{\partial \alpha} \right|_{\alpha=\lambda}$$

Then we have

$$\frac{i}{4\pi^2}\frac{(-i)^{m+n}}{(m-1)!(n-1)!}\oint_{\Gamma_1}\oint_{\Gamma_2} dk_1\, dk_2\, \frac{k_1^{\lambda+m-1}\ \ln(k_1) k_2^{\lambda+n-1}\ \ln(k_2)}{z-k_1-k_2}$$

$$= \left[\frac{i}{4\pi^2}\frac{(-1)^{m+n}}{(m-1)!(n-1)!}\right]\oint_{\Gamma_1} dk_1\, D_\alpha k^{\alpha+m-1}\oint_{\Gamma_2} dk_2\, \frac{D_\beta k_2^{\beta+n-1}\ \ln(k_2)}{z-k_1-k_2}$$

$$= [\ ] D_\alpha\, D_\beta \oint_{\Gamma_1} dk_1\, \frac{k_1^{\alpha+m-1}}{(z-k_1)^{1-\beta-n}}$$

$$\times\, 2i \sin\pi(\beta+n-1)\Gamma(\beta+n)\Gamma(1-\beta-n)$$

$$= 2\pi i[\ ]\, D_\alpha\, D_\beta \oint_{\Gamma_1} dk_1\, \frac{k_1^{\alpha+m-1}}{(z-k_1)^{1-\beta-n}}$$

$$= 2\pi i[\ ] D_\alpha\, D_\beta\, \frac{\Gamma(\alpha+m)\ \Gamma(1-\alpha-m\ \beta-n)}{\Gamma(1-\beta-n)}$$

$$\times\, 2i \sin\pi(\alpha+m-1)\ z^{\alpha+\beta+m+n-1}$$

$$= 4\pi[\ ]\, D_\alpha\, D_\beta\, \frac{\Gamma(\alpha+m)\Gamma(\beta+n)\sin\pi\alpha\sin\pi\beta}{\Gamma(\alpha+\beta+n+m)\sin\pi(\alpha+\beta)}\, z^{\alpha+\beta+m+n-1}$$

$$= -\frac{1}{\pi}\frac{(-i)^{m+n-1}}{(m+n-1)!}\, D_\alpha\, D_\beta \left\{\frac{\sin\pi\alpha\sin\pi\beta}{\sin\pi(\alpha+\beta)}\, z^{\alpha+\beta+m+n-1}\right\} \tag{5.17}$$

where we have used the fact that any derivative D_α or D_β acting on a Γ function will lead to a null result in (5.17) through the substitutions $\alpha = \lambda$, $\beta = \lambda$, $\lambda \to 0$. Now the derivatives in (5.17) give rise essentially to two types of terms. The two derivatives acting on the trigonometric functions give rise to a pole term (in λ). If one takes a derivative of the trigonometric functions and a derivative of $z^{\alpha+\beta}$, a constant term is obtained. For the term $D_\alpha\, D_\beta\, z^{\alpha+\beta}$ the limit $\lambda \to 0$ of the trigonometric functions is zero. Thus we get

$$(5.17) = -\frac{(-i)^{m+n-1}}{(m+n-1)!}\, z^{m+n-1}\left\{\frac{1}{4}\frac{1}{\lambda}\, z^{2\lambda}+\frac{1}{2}\ln(z)\right\}$$

The second and third terms of (5.16) have the same contribution, and can be evaluated by a similar procedure. This contribution is

$$\frac{1}{8\pi}\frac{(-i)^{m+n-2}}{(m-1)!(n-1)!}\oint_{\Gamma_1}\oint_{\Gamma_2} dk_1\, dk_2\, \frac{k_1^{\lambda+m-1}\ \ln(k_1) k_2^{\lambda+n-1}\epsilon(k_2)}{z-k_1-k_2}$$

$$= \frac{(-i)^{m+n}}{(m+n-1)!}\,\frac{\pi}{4}\,\epsilon(z) z^{m+n-1} \tag{5.18}$$

According to (5.17) and (5.18), we finally get

$$(5.16) = \frac{(-i)^{m+n}}{(m+n-1)!} z^{m+n-1}\left\{\frac{i}{4}\frac{1}{\lambda} z^{2\lambda} + \frac{i}{2}\ln(z) + \frac{\pi}{2}\epsilon(z)\right\}$$

$$= \frac{(-i)^{m+n}}{(m+n-1)!} z^{m+n-1}\left\{\frac{i}{4}\frac{1}{\lambda}[1 + 2\lambda \ln(z)] + \frac{i}{2}\ln(z) + \frac{\pi}{2}\epsilon(z)\right\}$$

$$- \frac{(-i)^{m+n}\pi}{(m+n-1)!} z^{m+n-1}\left\{-\frac{1}{\pi i}\ln(z) + \frac{1}{2}\epsilon(z)\right\} \quad (5.19)$$

The λ-independent term is recognized to be $\check{x}^{-m-n}$ [cf. (5.15)]. The pole term is equivalent to zero according to property III of Section 3.

(v) Finally, we give a physical example. We consider a massless scalar $(\lambda/4!)\phi^4(x)$ theory in four dimensions. For this theory we shall evaluate the self-energy Green function.

The propagator for the field $\phi(x)$ is [9]

$$\Delta(x) = [-4\pi^2(u^2 - i0)]^{-1} \quad (5.20)$$

According to Eqs. (A.5)–(A.10) of the Appendix we can write

$$(u^2 - i0)^{-1} = (2x_0)^{-1}[(x_0 - r)^{-1} + (x_0 + r)^{-1}]$$
$$+ (2r)^{-1}[\delta(x_0 - r) + \delta(x_0 + r)] + C\delta(x_0 - r)\delta(x_0 + r) \quad (5.21)$$

where C is an arbitrary constant appearing in the definition of some distributions (ref. 9, Sections 8.8, 8.9; see also Appendix).

And using the results of (i)–(iv), it is easy show that

$$(u^2 - i0)^{-1}(u^2 - i0)^{-1} = (u^2 - i0)^{-2}$$

Then, we have for the self-energy

$$\Sigma(x) = (\Delta(x))^2 = \frac{1}{16\pi^4}(u^2 - i0)^{-2} \quad (5.22)$$

where $(u^2 - i0)^{-2}$ is defined in ref. 9, Sections 8.8, 8.9.

6. DISCUSSION

When we use the perturbative development in quantum field theory, we have to deal with products of distributions in configuration space or with convolutions in the Fourier-transformed p space. Unfortunately, products or convolutions (of distributions) are in general ill-defined quantities. However, in physical applications one introduces some "regularization" scheme which allows us to give sense to divergent integrals. Among these procedures is the dimensional regularization method [14, 15], which essentially consists in the separation of the volume element $d^\nu p$ into an angular factor $d\Omega$ and a radial factor $p^{\nu-1}\, dp$. First the angular integration is carried out and then the number of dimensions ν is taken as a free parameter. It can be adjusted to give a convergent integral which is an analytic function of ν.

Our formula (4.7) is similar to the expression one obtains with dimensional regularization. However, the parameter λ is completely independent of any dimensional into interpretation.

All ultradistributions provide integrands [in (4.7)] that are analytic functions along the integration path. The parameter λ enables us to control the possible tempered asymptotic behavior [cf. Eq. (3.9)]. The existence of a region of analyticity for λ and a subsequent continuation to the point of interest [8] defines the convolution product.

Those properties show that tempered ultradistributions provide an appropriate framework for applications to physics. Furthermore, they can "absorb" arbitrary polynomials, thanks to Eq. (3.10), a property that is interesting for renormalization theory [See, for example, the elimination of the pole term in (5.19).] Consequently, we began this paper with a summary of the main characteristics of tempered ultradistributions and their Fourier-transformed distributions of the exponential type.

APPENDIX. DEFINITIONS

From ref. 8 we quote the formula

$$\begin{aligned}\mathcal{B}(\lambda, \mu) = &\int_0^{1/2} dx\, x^{\lambda-1}\left[(1-x)^{\mu-1} - \sum_{r=0}^{k-1}(-1)^r \frac{\Gamma(\mu)x^r}{r!\Gamma(\mu-r)}\right]\\ &+ \int_{1/2}^{1} dx\,(1-x)^{\mu-1}\left[x^{\lambda-1} - \sum_{r=0}^{s-1}(-1)^r \frac{\Gamma(\lambda)(1-x)^r}{r!\Gamma(\lambda-r)}\right]\\ &+ \sum_{r=0}^{k-1} \frac{(-1)^r\Gamma(\mu)}{2^{r+\lambda} r!\Gamma(\mu-r)(r+\lambda)} + \sum_{r=0}^{s-1} \frac{(-1)^r\Gamma(\lambda)}{2^{r+\mu} r!\Gamma(\lambda-r)(r+\mu)}\end{aligned} \tag{A.1}$$

valid for Re $\lambda > -k$, Rc $\mu > -s$, where k and s are positive integers. From ref. 13 we get

$$\mathcal{B}(\lambda, \mu) = \frac{\Gamma(\lambda)\Gamma(\mu)}{\Gamma(\lambda+\mu)} \tag{A.2}$$

$$\Gamma(\lambda) = \int_0^\infty dt\, t^{\lambda-1} e^{-t} \tag{A.3}$$

$$\Gamma(\lambda)\Gamma(1 - \lambda) = \frac{\pi}{\sin \pi\lambda} \tag{A.4}$$

From ref. 9 we have

$$\begin{aligned}\delta^{(m)}(u^2) &= \delta^{(m)} (x^0 + r)(x^0 - r)^{-m-1} \operatorname{sgn}(x^0 - r) \\ &= + \delta^{(m)}(x^0 - r)(x^0 + r)^{-m-1} \operatorname{sgn}(x^0 + r)\end{aligned} \tag{A.5}$$

where

$$u^2 = x_0^2 - x_1^2 - \cdots - x_{n-1}^2 \tag{A.6}$$

$$r^2 = x_1^2 + x_2^2 + \cdots + x_{n-1}^2 \tag{A.7}$$

$$(u^2 \pm i0)^{-m} = u^{-2m} \pm \frac{(-1)^m}{(m-1)!} i\pi\, \delta^{(m-1)}(u^2) \tag{A.8}$$

$$x^{-m} \operatorname{sgn}(x) = \frac{(-1)^{m-1}}{(m-1)!} \{|x|^{-1}\}^{(m-1)} \tag{A.9}$$

$$|x|^{-1} = \{\operatorname{sgn}(x) \ln|x|\}' + C\, \delta(x) \tag{A.10}$$

where C is an arbitrary constant.

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